CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS

BY

GEORG CANTOR

TRANSLATED, AND PROVIDED WITH AN INTRODUCTION AND NOTES, BY

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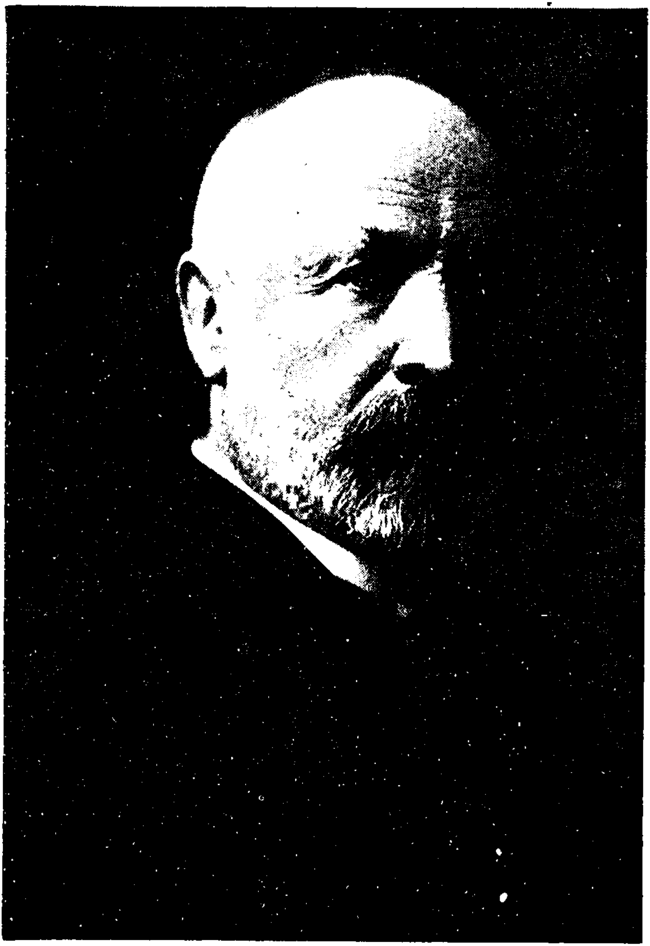

GEORG CANTOR (1845–1918)

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PREFACE

THIS volume contains a translation of the two very important memoirs of Georg Cantor on transfinite numbers which appeared in the *Mathematische Annalen* for 1895 and 1897* under the title: "Beiträge zur Begründung der transfiniten Mengenlehre." It seems to me that, since these memoirs are chiefly occupied with the investigation of the various transfinite cardinal and ordinal numbers and not with investigations belonging to what is usually described as "the theory of aggregates" or "the theory of sets" (*Mengenlehre*, *théorie des ensembles*), —the elements of the sets being real or complex numbers which are imaged as geometrical "points" in space of one or more dimensions,—the title given to them in this translation is more suitable.

These memoirs are the final and logically purified statement of many of the most important results of the long series of memoirs begun by Cantor in 1870. It is, I think, necessary, if we are to appreciate the full import of Cantor's work on transfinite numbers, to have thought through and to bear in mind Cantor's earlier researches on the theory of point-aggregates. It was in these researches that the need for the

* Vol. xlvi, 1895, pp. 481–512; vol. xlix, 1897, pp. 207–246.

transfinite numbers first showed itself, and it is only by the study of these researches that the majority of us can annihilate the feeling of arbitrariness and even insecurity about the introduction of these numbers. Furthermore, it is also necessary to trace backwards, especially through Weierstrass, the course of those researches which led to Cantor's work. I have, then, prefixed an Introduction tracing the growth of parts of the theory of functions during the nineteenth century, and dealing, in some detail, with the fundamental work of Weierstrass and others, and with the work of Cantor from 1870 to 1895. Some notes at the end contain a short account of the developments of the theory of transfinite numbers since 1897. In these notes and in the Introduction I have been greatly helped by the information that Professor Cantor gave me in the course of a long correspondence on the theory of aggregates which we carried on many years ago.

The philosophical revolution brought about by Cantor's work was even greater, perhaps, than the mathematical one. With few exceptions, mathematicians joyfully accepted, built upon, scrutinized, and perfected the foundations of Cantor's undying theory; but very many philosophers combated it. This seems to have been because very few understood it. I hope that this book may help to make the subject better known to both philosophers and mathematicians.

The three men whose influence on modern pure mathematics—and indirectly modern logic and the philosophy which abuts on it—is most marked are Karl Weierstrass, Richard Dedekind, and Georg Cantor. A great part of Dedekind's work has developed along a direction parallel to the work of Cantor, and it is instructive to compare with Cantor's work Dedekind's *Stetigkeit und irrationale Zahlen* and *Was sind und was sollen die Zahlen?*, of which excellent English translations have been issued by the publishers of the present book.*

There is a French translation † of these memoirs of Cantor's, but there is no English translation of them. For kind permission to make the translation, I am indebted to Messrs B. G. Teubner of Leipzig and Berlin, the publishers of the *Mathematische Annalen*.

PHILIP E. B. JOURDAIN.

* *Essays on the Theory of Numbers* (I, *Continuity and Irrational Numbers*; II, *The Nature and Meaning of Numbers*), translated by W. W. Beman, Chicago, 1901. I shall refer to this as *Essays on Number*.

† By F. Marotte, *Sur les fondements de la théorie des ensembles transfinis*, Paris, 1899.

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CONTRIBUTIONS TO THE FOUNDING OF THE THEORY OF TRANSFINITE NUMBERS

INTRODUCTION

I

If it is safe to trace back to any single man the origin of those conceptions with which pure mathematical analysis has been chiefly occupied during the nineteenth century and up to the present time, we must, I think, trace it back to Jean Baptiste Joseph Fourier (1768–1830). Fourier was first and foremost a physicist, and he expressed very definitely his view that mathematics only justifies itself by the help it gives towards the solution of physical problems, and yet the light that was thrown on the general conception of a function and its "continuity," of the "convergence" of infinite series, and of an integral, first began to shine as a result of Fourier's original and bold treatment of the problems of the conduction of heat. This it was that gave the impetus to the formation and development of the theories of functions. The broad-minded physicist will approve of this refining development of the mathematical methods which arise from physical conceptions when he reflects that mathematics is a wonderfully powerful and economically contrived means of dealing logically and conveniently with an immense complex of data, and that we cannot be sure of the logical soundness of our methods and results until we make everything about them quite definite. The pure mathematician knows that pure mathematics has an end in itself which is more allied with philosophy. But we have not to justify pure mathematics here: we have only to point out its origin in physical conceptions. But we have also pointed out that physics can justify even the most modern developments of pure mathematics.

II

During the nineteenth century, the two great branches of the theory of functions developed and gradually separated. The rigorous foundation of the results of Fourier on trigonometrical series, which was given by Dirichlet, brought forward as subjects of investigation the general conception of a (one-valued) function of a real variable and the (in particular, trigonometrical) development of functions. On the other hand, Cauchy was gradually led to recognize the importance of what was subsequently seen to be the more special conception of function of a complex variable; and, to a great extent independently of Cauchy, Weierstrass built up his theory of analytic functions of complex variables.

These tendencies of both Cauchy and Dirichlet combined to influence Riemann; his work on the theory of functions of a complex variable carried on and greatly developed the work of Cauchy, while the intention of his "Habilitationsschrift" of 1854 was to generalize as far as possible Dirichlet's partial solution of the problem of the development of a function of a real variable in a trigonometrical series.

Both these sides of Riemann's activity left a deep impression on Hankel. In a memoir of 1870, Hankel attempted to exhibit the theory of functions of a real variable as leading, of necessity, to the restrictions and extensions from which we start in Riemann's theory of functions of a complex variable; and yet Hankel's researches entitle him to be called the founder of the independent theory of functions of a real variable. At about the same time, Heine initiated, under the direct influence of Riemann's "Habilitationsschrift," a new series of investigations on trigonometrical series.

Finally, soon after this, we find Georg Cantor both studying Hankel's memoir and applying to theorems on the uniqueness of trigonometrical developments those conceptions of his on irrational numbers and the "derivatives" of point-aggregates or number-aggregates which developed from the rigorous treatment of such fundamental questions given by Weierstrass at Berlin in the introduction to his lectures on analytic functions. The theory of point-aggregates soon became an independent theory of great importance, and finally, in 1882, Cantor's "transfinite numbers" were defined independently of the aggregates in connexion with which they first appeared in mathematics.

III

The investigations * of the eighteenth century on the problem of vibrating cords led to a controversy for the following reasons. D'Alembert maintained that the arbitrary functions in his general integral of the partial differential equation to which this problem led were restricted to have certain properties which assimilate them to the analytically representable functions then known, and which would prevent their course being completely arbitrary at every point. Euler, on the other hand, argued for the admission of certain of these "arbitrary" functions into analysis. Then Daniel Bernoulli produced a solution in the form of an infinite trigonometrical series, and claimed, on certain physical grounds, that this solution was as general as d'Alembert's. As Euler pointed out, this was so only if any arbitrary † function $\phi(x)$ were developable in a series of the form

* *Cf.* the references given in my papers in the *Archiv der Mathematik und Physik*, 3rd series, vol. x, 1906, pp. 255–256, and *Isis*, vol. i, 1914, pp. 670–677. Much of this Introduction is taken from my account of "The Development of the Theory of Transfinite Numbers" in the above-mentioned *Archiv*, 3rd series, vol. x, pp. 254–281; vol. xiv, 1909, pp. 289–311; vol. xvi, 1910, pp. 21–43; vol. xxii, 1913, pp. 1–21.

† The arbitrary functions chiefly considered in this connexion by Euler were what he called "discontinuous" functions. This word does not mean what we now mean (after Cauchy) by it. *Cf.* my paper in *Isis*, vol. i, 1914, pp. 661–703.

$$\phi(x)=\sum_\nu a_\nu \sin\frac{\nu\pi x}{l}.$$

That this was, indeed, the case, even when $\phi(x)$ is not necessarily developable in a power-series, was first shown by Fourier, who was led to study the same mathematical problem as the above one by his researches, the first of which were communicated to the French Academy in 1807, on the conduction of heat. To Fourier is due also the determination of the coefficients in trigonometric series,

$$\phi(x)=\tfrac{1}{2}b_0+b_1\cos x+b_2\cos 2x+\ldots$$
$$+a_1\sin x+a_2\sin 2x+\ldots,$$

in the form

$$b_\nu=\frac{1}{\pi}\int_{-\pi}^{+\pi}\phi(a)\cos\nu a\,da,\qquad a_\nu=\frac{1}{\pi}\int_{-\pi}^{+\pi}\phi(a)\sin\nu a\,da.$$

This determination was probably independent of Euler's prior determination and Lagrange's analogous determination of the coefficients of a *finite* trigonometrical series. Fourier also gave a geometrical proof of the convergence of his series, which, though not formally exact, contained the germ of Dirichlet's proof.

To Peter Gustav Lejeune-Dirichlet (1805–1859) is due the first exact treatment of Fourier's series.* He expressed the sum of the first n terms of the series by a definite integral, and proved that the

* "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données," *Journ. für Math.*, vol. iv, 1829, pp. 157–169; *Ges. Werke*, vol. i, pp. 117–132.

limit, when n increases indefinitely, of this integral is the function which is to be represented by the trigonometrical series, provided that the function satisfies certain conditions. These conditions were somewhat lightened by Lipschitz in 1864.

Thus, Fourier's work led to the contemplation and exact treatment of certain functions which were totally different in behaviour from algebraic functions. These last functions were, before him, tacitly considered to be the type of all functions that can occur in analysis. Henceforth it was part of the business of analysis to investigate such non-algebraoid functions.

In the first few decades of the nineteenth century there grew up a theory of more special functions of an imaginary or complex variable. This theory was known, in part at least, to Carl Friedrich Gauss (1777–1855), but he did not publish his results, and so the theory is due to Augustin Louis Cauchy (1789–1857).* Cauchy was less far-sighted and penetrating than Gauss, the theory developed slowly, and only gradually were Cauchy's prejudices against "imaginaries" overcome. Through the years from 1814 to 1846 we can trace, first, the strong influence on Cauchy's conceptions of Fourier's ideas, then the quickly increasing unsusceptibility to the ideas of others, coupled with the extraordinarily prolific nature of this narrow-minded genius. Cauchy appeared to take pride in the production of memoirs

* *Cf.* Jourdain, "The Theory of Functions with Cauchy and Gauss," *Bibl. Math.* (3), vol. vi, 1905, pp. 190–207.

at each weekly meeting of the French Academy; and it was partly, perhaps, due to this circumstance that his works are of very unequal importance. Besides that, he did not seem to perceive even approximately the immense importance of the theory of functions of a complex variable which he did so much to create. This task remained for Puiseux, Briot and Bouquet, and others, and was advanced in the most striking manner by Georg Friedrich Bernhard Riemann (1826–1866).

Riemann may have owed to his teacher Dirichlet his bent both towards the theory of potential—which was the chief instrument in his classical development (1851) of the theory of functions of a complex variable—and that of trigonometrical series. By a memoir on the representability of a function by a trigonometrical series, which was read in 1854 but only published after his death, he not only laid the foundations for all modern investigations into the theory of these series, but inspired Hermann Hankel (1839–1873) to the method of researches from which we can date the theory of functions of a real variable as an independent science. The motive of Hankel's research was provided by reflexion on the foundations of Riemann's theory of functions of a complex variable. It was Hankel's object to show how the needs of mathematics compel us to go beyond the most general conception of a function, which was implicitly formulated by Dirichlet, to introduce the complex variable, and finally to reach that conception from which Riemann started in his inaugural dissertation. For this purpose Hankel began his "Untersuchungen über die unendlich oft oscillirenden und unstetigen Functionen; ein Beitrag zur Feststellung des Begriffes der Function überhaupt" of 1870 by a thorough examination of the various possibilities contained in Dirichlet's conception.

Riemann, in his memoir of 1854, started from the general problem of which Dirichlet had only solved a particular case: If a function is developable in a trigonometrical series, what results about the variation of the value of the function (that is to say, what is the most general way in which it can become discontinuous and have maxima and minima) when the argument varies continuously? The argument is a real variable, for Fourier's series, as Fourier had already noticed, may converge for real x's alone. This question was not completely answered, and, perhaps in consequence of this, the work was not published in Riemann's lifetime; but fortunately that part of it which concerns us more particularly, and which seems to fill, and more than fill, the place of Dirichlet's contemplated revision of the principles of the infinitesimal calculus, has the finality obtained by the giving of the necessary and sufficient conditions for the integrability of a function $f(x)$, which was a necessary preliminary to Riemann's investigation. Thus, Riemann was led to give the process of integration a far wider meaning than that contemplated by Cauchy or even Dirichlet, and Riemann constructed an integrable function which becomes discontinuous an infinity of times between any two limits, as close together as wished, of the independent variable, in the following manner :—If, where x is a real variable, (x) denotes the (positive or negative) excess of x over the nearest integer, or zero if x is midway between two integers, (x) is a one-valued function of x with discontinuities at the points $x=n+\frac{1}{2}$, where n is an integer (positive, negative, or zero), and with $\frac{1}{2}$ and $-\frac{1}{2}$ for upper and lower limits respectively. Further, (νx), where ν is an integer, is discontinuous at the points $\nu x=n+\frac{1}{2}$ or $x=\frac{1}{\nu}(n+\frac{1}{2})$. Consequently, the series

$$f(x)=\sum_{\nu=1}^{\infty}\frac{(\nu x)}{\nu^2},$$

where the factor $1/\nu^2$ is added to ensure convergence for all values of x, may be supposed to be discontinuous for all values of x of the form $x=p/2n$, where p is an odd integer, relatively prime to n. It was this method that was, in a certain respect, generalized by Hankel in 1870. In Riemann's example appeared an analytical expression—and therefore a "function" in Euler's sense—which, on account of its manifold singularities, allowed of no such general properties as Riemann's "functions of a complex variable," and Hankel gave a method, whose principles were suggested by this example, of forming analytical expressions with singularities at every rational point. He was thus led to state, with some reserve, that every "function" in Dirichlet's sense is also a "function" in Euler's sense.

The greatest influence on Georg Cantor seems, however, not to have been that exercised by Riemann, Hankel, and their successors—though the work of these men is closely connected with some parts of Cantor's work,—but by Weierstrass, a contemporary of Riemann's, who attacked many of the same problems in the theory of analytic functions of complex variables by very different and more rigorous methods.

IV

Karl Weierstrass (1815–1897) has explained, in his address delivered on the occasion of his entry into the Berlin Academy in 1857, that, from the time (the winter of 1839–1840) when, under his teacher Gudermann, he made his first acquaintance with the theory of elliptic functions, he was powerfully attracted by this branch of analysis. "Now, Abel, who was accustomed to take the highest standpoint in any part of mathematics, established a theorem which comprises all those transcendents which arise from the integration of algebraic differentials, and has the same signification for these as Euler's integral has for elliptic functions . . . ; and Jacobi succeeded in demonstrating the existence of periodic functions of *many arguments*, whose fundamental properties are established in Abel's theorem, and by means of which the true meaning and real essence of this theorem could be judged. Actually to represent, and to investigate the properties of these magnitudes of a totally new kind, of which analysis has as yet no example, I regarded as one of the principal problems of mathematics, and, as soon as I clearly recognized the meaning and significance of this problem, resolved to devote myself to it. Of course it would have been foolish even to think of the solution of such a problem without having prepared myself by a thorough study of the means and by busying myself with less difficult problems."

With the *ends* stated here of Weierstrass's work we are now concerned only incidentally: it is the *means*—the "thorough study" of which he spoke—which has had a decisive influence on our subject in common with the theory of functions. We will, then, pass over his early work—which was only published in 1894—on the theory of analytic functions, his later work on the same subject, and his theory of the Abelian functions, and examine his immensely important work on the foundations of arithmetic, to which he was led by the needs of a rigorous theory of analytic functions.

We have spoken as if the ultimate aim of Weierstrass's work was the investigation of Abelian functions. But another and more philosophical view was expressed in his introduction to a course of lectures delivered in the summer of 1886 and preserved by Gosta Mittag-Leffler*: "In order to penetrate into mathematical science it is indispensable that we should occupy ourselves with individual

* "Sur les fondements arithmétiques de la théorie des fonctions d'après Weierstrass," *Congrès des Mathématiques à Stockholm*, 1909, p. 10.

problems which show us its extent and constitution. But the final object which we must always keep in sight is the attainment of a sound judgment on the foundations of science."

In 1859, Weierstrass began his lectures on the theory of analytic functions at the University of Berlin. The importance of this, from our present point of view, lies in the fact that he was naturally obliged to pay special attention to the systematic treatment of the theory, and consequently, to scrutinize its foundations.

In the first place, one of the characteristics of Weierstrass's theory of functions is the abolition of the method of complex integration of Cauchy and Gauss which was used by Riemann; and, in a letter to H. A. Schwarz of October 3, 1875, Weierstrass stated his belief that, in a systematic foundation, it is better to dispense with integration, as follows:—

". . . The more I meditate upon the principles of the theory of functions,—and I do this incessantly,—the firmer becomes my conviction that this theory must be built up on the foundation of algebraic truths, and therefore that it is not the right way to proceed conversely and make use of the *transcendental* (to express myself briefly) for the establishment of simple and fundamental algebraic theorems; however attractive may be, for example, the considerations by which Riemann discovered so many of the most important properties of algebraic functions. That to the discoverer, *quâ* discoverer, every route is permissible, is, of course, self-evident; I am only thinking of the systematic establishment of the theory."

In the second place, and what is far more important than the question of integration, the systematic treatment, *ab initio*, of the theory of analytic functions led Weierstrass to profound investigations in the principles of arithmetic, and the great result of these investigations—his theory of irrational numbers—has a significance for all mathematics which can hardly be overrated, and our present subject may truly be said to be almost wholly due to this theory and its development by Cantor.

In the theory of analytic functions we often have to use the theorem that, if we are given an infinity of points of the complex plane in any bounded region of this plane, there is at least one point of the domain such that there is an infinity of the given points in each and every neighbourhood round it and including it. Mathematicians used to express this by some such rather obscure phrase as: "There is a point near which some of the given points are infinitely near to one another." If we apply, for the proof of this, the method which seems naturally to suggest itself, and which consists in successively halving the region or one part of the region which contains an infinity of points,* we arrive at what is required,—namely, the conclusion that there is a point such that there is another point in *any* neigh-

* This method was first used by Bernard Bolzano in 1817.